Phase Structure of Nambu–Jona-Lasinio Model at Finite Isospin Density

Lianyi He and Pengfei Zhuang
Physics Department, Tsinghua University, Beijing 100084, China
(February 2, 2008)

In the frame of flavor $SU(2)$ Nambu–Jona-Lasinio model with $U_A(1)$ breaking term we found that, the structure of two chiral phase transition lines does not exist at low isospin density in real world, and the critical isospin chemical potential for pion superfluidity is exactly the pion mass in the vacuum.

pacs numbers: 11.10.Wx, 12.38.-t 25.75.Nq

I. INTRODUCTION

It is generally believed that there exists a rich phase structure of Quantum Chromodynamics (QCD) at finite temperature and baryon density, for instance, the deconfinement process from hadron gas to quark-gluon plasma, the transition from chiral symmetry breaking phase to the symmetry restoration phase [1], and the color superconductivity [2] at low temperature and high baryon density. Recently, the study on the QCD phase structure is extended to finite isospin density. The physical motivation to study isospin spontaneous breaking and the corresponding pion superfluidity is related to the investigation of compact stars, isospin asymmetric nuclear matter and heavy ion collisions at intermediate energies.

While the perturbation theory of QCD can describe well the properties of the new phases at high temperatures and/or high densities, the study on the phase structure at moderate (baryon or isospin) density depends on lattice QCD calculation and effective models with QCD symmetries. While there is not yet precise lattice result at finite baryon density due to the Fermion sign problem [3], it is in principle no problem to do lattice simulation at finite isospin density [4]. It is found [5] that the critical isospin chemical potential for pion condensation is about the pion mass in the vacuum, $\mu_I^c \simeq m_\pi$. The QCD phase structure at finite isospin density is also investigated in many low energy effective models, such as chiral perturbation theory [4,6,7], Nambu–Jona-Lasinio (NJL) model [11–13],ladder QCD [8], strong coupling lattice QCD [10] and random matrix method [9].

One of the models that enables us to see directly how the dynamic mechanisms of chiral symmetry breaking and restoration operate is the NJL model [14] applied to quarks [15]. Within this model, one can obtain the hadronic mass spectrum and the static properties of mesons remarkably well [15,16], and the chiral phase transition line [15–17] in the temperature and baryon chemical potential $(T-\mu_B)$ plane is very close to the one calculated with lattice QCD. Recently, this model is also used to investigate the color superconductivity at moderate baryon density [18–22]. In the study at finite isospin density, it is predicted [11] in this model that the chiral phase transition line in $T-\mu_B$ plane splits into two branches. This phenomena is also found in random matrix method [9] and ladder QCD [8]. Since the NJL Lagrangian used in [11] does not contain the determinant term which breaks the $U_A(1)$ symmetry and leads to reasonable meson mass splitting, it is pointed out [13] that the presence of the $U_A(1)$ breaking term will cancel the structure of the two chiral phase transition lines, if the coupling constant describing the $U_A(1)$ breaking term is large enough. However, this coupling constant is considered as a free parameter and not yet determined in [13]. Another problem in the NJL calculation at finite isosipn density is that the critical isospion chemical potential for pion condensation $\mu_I^c = m_\pi$ is not recovered in the model [12]. In this letter we will focus on these two problems in the frame of NJL model with $U_A(1)$ breaking term. We hope to derive the critical isospin chemical potential exactly and try to fix the chiral structure by fitting the meson masses in the vacuum.

The letter is organized as follows. We present at finite temperature and baryon and isospin densities the chiral and pion condensates in mean field approximation and meson masses in Random Phase Approximation (RPA) in Section 2, determine the coupling constant of the $U_A(1)$ breaking term in Section 3, and then analytically prove the relation $\mu_I^c = m_\pi$ in Section 4. We conclude in Section 5.

II. NJL MODEL AT FINITE TEMPERATURE AND BARYON AND ISOSPIN DENSITIES

We start with the flavor $SU(2)$ NJL model defined by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_0 + \mu\gamma_0)\psi + \mathcal{L}_{int} \ , \tag{1}$$

where m_0 is the current quark mass, μ the chemical potential matrix in flavor space,

$$\mu = \begin{pmatrix} \mu_u & 0 \\ 0 & \mu_d \end{pmatrix} = \begin{pmatrix} \frac{\mu_B}{N_c} + \frac{\mu_I}{2} & 0 \\ 0 & \frac{\mu_B}{N_c} - \frac{\mu_I}{2} \end{pmatrix} \tag{2}$$

with μ_B and μ_I being the baryon and isospin chemical potential, respectively, and the interaction part includes [13] the normal four Fermion couplings corresponding to scalar mesons σ, a_0, a_+ and a_- and pseudoscalar mesons $\eta\prime, \pi_0, \pi_+$ and π_- excitations, and the 't-Hooft [23] determinant term for $U_A(1)$ breaking,

$$\begin{aligned} \mathcal{L}_{int} &= \frac{G}{2}\sum_{a=0}^{3}\left[(\bar\psi\tau_a\psi)^2 + (\bar\psi i\gamma_5\tau_a\psi)^2\right] + \frac{K}{2}\left[\det\bar\psi(1+\gamma_5)\psi + \det\bar\psi(1-\gamma_5)\psi\right] \\ &= \frac{1}{2}(G+K)\left[(\bar\psi\psi)^2 + (\bar\psi i\gamma_5\vec\tau\psi)^2\right] + \frac{1}{2}(G-K)\left[(\bar\psi\vec\tau\psi)^2 + (\bar\psi i\gamma_5\psi)^2\right] \ . \end{aligned} \tag{3}$$

For $K=0$ and $m_0=0$, the Lagrangian is invariant under $U_B(1)\bigotimes U_A(1)\bigotimes SU_V(2)\bigotimes SU_A(2)$ transformations, but for $K\neq 0$, the symmetry is reduced to $U_B(1)\bigotimes SU_V(2)\bigotimes SU_A(2)$ and the $U_A(1)$ breaking leads to σ and a mass splitting and π and $\eta\prime$ mass splitting. If $G=K$, we come back to the standard NJL model [15] with only σ, π_0, π_+ and π_- mesons.

We introduce the quark condensates

$$\sigma_u = \langle\bar u u\rangle \ , \qquad \sigma_d = \langle\bar d d\rangle \ , \tag{4}$$

or equivalently the σ and a_0 condensates

$$\begin{aligned} \sigma &= \langle\bar\psi\psi\rangle = \langle\bar u u + \bar d d\rangle = \sigma_u + \sigma_d \ , \\ a_0 &= \langle\bar\psi\tau_3\psi\rangle = \langle\bar u u - \bar d d\rangle = \sigma_u - \sigma_d \ , \end{aligned} \tag{5}$$

and the pion condensate

$$\frac{\pi}{\sqrt{2}} = \langle\bar\psi i\gamma_5\tau_+\psi\rangle = \langle\bar\psi i\gamma_5\tau_-\psi\rangle = \frac{1}{\sqrt{2}}\langle\bar\psi i\tau_1\gamma_5\psi\rangle \ , \tag{6}$$

where we have chosen the pion condensate to be real. The quark condensate and pion condensate are, respectively, the order parameter of chiral phase transition and pion superfluidity. By separating each Lorentz scalar in the Lagrangian (3) into the classical condensate and the quantum fluctuation, and keeping only the linear terms in the fluctuations, one obtains the Lagrangian in mean field approximation,

$$\mathcal{L}_{mf} = \bar\psi\mathcal{S}_{mf}^{-1}\psi - G(\sigma_u^2+\sigma_d^2) - 2K\sigma_u\sigma_d - \frac{G+K}{2}\pi^2 \ , \tag{7}$$

where $\mathcal{S}_{mf}^{-1}$ is the inverse of the mean field quark propagator, in momentum space it reads

$$\mathcal{S}_{mf}^{-1}(k) = \begin{pmatrix} \gamma^\mu k_\mu + \mu_u\gamma_0 - M_u & i(G+K)\pi\gamma_5 \\ i(G+K)\pi\gamma_5 & \gamma^\mu k_\mu + \mu_d\gamma_0 - M_d \end{pmatrix} \tag{8}$$

with the effective quark masses

$$\begin{aligned} M_u &= m_0 - 2G\sigma_u - 2K\sigma_d \ , \\ M_d &= m_0 - 2G\sigma_d - 2K\sigma_u \ . \end{aligned} \tag{9}$$

The thermodynamic potential of the system in mean field approximation can be expressed in terms of the effective quark propagator,

$$\Omega(T,\mu_B,\mu_I;\sigma_u,\sigma_d,\pi) = G(\sigma_u^2+\sigma_d^2) + 2K\sigma_u\sigma_d + \frac{G+K}{2}\pi^2 - \frac{T}{V}\ln\det\mathcal{S}_{mf}^{-1}(k) \ . \tag{10}$$

The condensates σ_u, σ_d and π as functions of temperature and baryon and isospin chemical potentials are determined by the minimum thermodynamic potential,

$$\frac{\partial\Omega}{\partial\sigma_u}=0, \quad \frac{\partial\Omega}{\partial\sigma_d}=0, \quad \frac{\partial\Omega}{\partial\pi}=0\ . \tag{11}$$

It is easy to see from the chemical potential matrix and the quark propagator matrix that for $\mu_B=0$ or $\mu_I=0$ the gap equations for σ_u and σ_d are symmetric, and one has

$$\begin{aligned} &\sigma_u=\sigma_d=\frac{\sigma}{2}\ , \qquad a_0=0\ , \\ &M_u=M_d=M_q=m_0-(G+K)\sigma\ . \end{aligned} \tag{12}$$

The quark propagator in flavor space can be formally expressed as

$$\mathcal{S}_{mf}(k)=\begin{pmatrix} \mathcal{S}_{uu}(k) & \mathcal{S}_{ud}(k) \\ \mathcal{S}_{du}(k) & \mathcal{S}_{dd}(k) \end{pmatrix}\ . \tag{13}$$

From the comparison with the definition of the condensates (4) and (6), one can express the condensates in terms of the matrix elements of $\mathcal{S}_{mf}$,

$$\begin{aligned} \sigma_u &= -\int\frac{d^4k}{(2\pi)^4}\mathbf{Tr}\left[i\mathcal{S}_{uu}(k)\right]\ , \\ \sigma_d &= -\int\frac{d^4k}{(2\pi)^4}\mathbf{Tr}\left[i\mathcal{S}_{dd}(k)\right]\ , \\ \pi &= \int\frac{d^4k}{(2\pi)^4}\mathbf{Tr}\left[\left(\mathcal{S}_{ud}(k)+\mathcal{S}_{du}(k)\right)\gamma_5\right]\ . \end{aligned} \tag{14}$$

where the trace is in color and spin space.

In the self-consistent mean field approximation, it is well known that the meson masses M_M as the bound states of the colliding quark-antiquark pairs are determined as the poles of the meson propagators in RPA at zero momentum,

$$1-(G+K)\Pi_M\left(k_0=M_M,\mathbf{k}=\mathbf{0}\right)=0 \tag{15}$$

for σ and π, and

$$1-(G-K)\Pi_M\left(k_0=M_M,\mathbf{k}=\mathbf{0}\right)=0 \tag{16}$$

for $\eta\prime$ and a, where Π_M is the meson polarization function

$$-i\Pi_M(k)=-\int\frac{d^4p}{(2\pi)^4}\mathbf{Tr}\left[\Gamma_M^* i\mathcal{S}_{mf}(p+k)\Gamma_M i\mathcal{S}_{mf}(p)\right]\ , \tag{17}$$

and Γ_M and Γ_M^* are the interaction vertexes

$$\Gamma_M=\begin{cases} 1 & M=\sigma \\ \tau_3 & M=a_0 \\ \tau_+ & M=a_+ \\ \tau_- & M=a_- \\ i\gamma_5 & M=\eta' \\ i\gamma_5\tau_3 & M=\pi_0 \\ i\gamma_5\tau_+ & M=\pi_+ \\ i\gamma_5\tau_- & M=\pi_- \end{cases}\ , \qquad \Gamma_M^*=\begin{cases} 1 & M=\sigma \\ \tau_3 & M=a_0 \\ \tau_- & M=a_+ \\ \tau_+ & M=a_- \\ i\gamma_5 & M=\eta' \\ i\gamma_5\tau_3 & M=\pi_0 \\ i\gamma_5\tau_- & M=\pi_+ \\ i\gamma_5\tau_+ & M=\pi_- \end{cases} \tag{18}$$

with $\tau_\pm=(\tau_1\pm i\tau_2)/\sqrt{2}$. Doing the trace in flavor and color spaces, one has

$$\begin{aligned} \Pi_{a_+}(k) &= 2N_c i\int\frac{d^4p}{(2\pi)^4}\mathbf{tr}\left[\mathcal{S}_{uu}(p+k)\mathcal{S}_{dd}(p)\right]\ , \\ \Pi_{a_-}(k) &= 2N_c i\int\frac{d^4p}{(2\pi)^4}\mathbf{tr}\left[\mathcal{S}_{dd}(p+k)\mathcal{S}_{uu}(p)\right]\ , \\ \Pi_{\pi_+}(k) &= -2N_c i\int\frac{d^4p}{(2\pi)^4}\mathbf{tr}\left[\gamma_5\mathcal{S}_{uu}(p+k)\gamma_5\mathcal{S}_{dd}(p)\right]\ , \end{aligned} \tag{19}$$

$$
\begin{aligned}
&\Pi_{\pi_-}(k) = -2N_c i\int \frac{d^4p}{(2\pi)^4}\mathbf{tr}\left[\gamma_5\mathcal{S}_{dd}(p+k)\gamma_5\mathcal{S}_{uu}(p)\right] \ , \\
&\Pi_\sigma(k) = \Pi_{a_0}(k) = N_c i\int \frac{d^4p}{(2\pi)^4}\mathbf{tr}\left[\mathcal{S}_{uu}(p+k)\mathcal{S}_{uu}(p)+\mathcal{S}_{ud}(p+k)\mathcal{S}_{du}(p)+\mathcal{S}_{du}(p+k)\mathcal{S}_{ud}(p)+\mathcal{S}_{dd}(p+k)\mathcal{S}_{dd}(p)\right] \ , \\
&\Pi_{\eta'}(k) = \Pi_{\pi_0}(k) \\
&= -N_c i\int \frac{d^4p}{(2\pi)^4}\mathbf{tr}\left[\gamma_5\mathcal{S}_{uu}(p+k)\gamma_5\mathcal{S}_{uu}(p)-\gamma_5\mathcal{S}_{ud}(p+k)\gamma_5\mathcal{S}_{du}(p)-\gamma_5\mathcal{S}_{du}(p+k)\gamma_5\mathcal{S}_{ud}(p)+\gamma_5\mathcal{S}_{dd}(p+k)\gamma_5\mathcal{S}_{dd}(p)\right] \ ,
\end{aligned}
$$

now the trace is taken only in spin space.

III. CHIRAL PHASE STRUCTURE

We now consider the QCD phase structure below the minimum isospin chemical potential μ_I^c for pion superfluidity. Since the pion condensate is zero, there is only chiral phase structure in this region.

The simple diagonal matrix $\mathcal{S}_{mf}^{-1}(k)$ in this region makes it easy to calculate the matrix elements of the effective quark propagator, they can be expressed explicitly as

$$
\begin{aligned}
\mathcal{S}_{uu}(k) &= \frac{\Lambda_+^u\gamma_0}{k_0-E_1}+\frac{\Lambda_-^u\gamma_0}{k_0+E_2} \ , \\
\mathcal{S}_{dd}(k) &= \frac{\Lambda_+^d\gamma_0}{k_0-E_3}+\frac{\Lambda_-^d\gamma_0}{k_0+E_4}, \\
\mathcal{S}_{ud}(k) &= \mathcal{S}_{du}(k) = 0 \ ,
\end{aligned}
\tag{20}
$$

with quasiparticle energies

$$
\begin{aligned}
&E_1 = E_u-\mu_u \ , \quad E_2 = E_u+\mu_u \ , \quad E_3 = E_d-\mu_d \ , \quad E_4 = E_d+\mu_d \ , \\
&E_u = \sqrt{\mathbf{k}^2+M_u^2} \ , \quad E_d = \sqrt{\mathbf{k}^2+M_d^2} \ ,
\end{aligned}
\tag{21}
$$

and energy projectors

$$
\Lambda_\pm^{u,d} = \frac{1}{2}\left(1\pm\frac{\gamma_0\left(\gamma\cdot\mathbf{k}+M_{u,d}\right)}{E_{u,d}}\right) \ . \tag{22}
$$

After performing the Matsubara frequency summation in (14), the gap equations determining the two quark condensates as functions of temperature and baryon and isospin chemical potentials read

$$
\begin{aligned}
\sigma_u &= -2N_c\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{M_u}{\sqrt{k^2+M_u^2}}\left(1-f(E_1)-f(E_2)\right) \ , \\
\sigma_d &= -2N_c\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{M_d}{\sqrt{k^2+M_d^2}}\left(1-f(E_3)-f(E_4)\right) \ ,
\end{aligned}
\tag{23}
$$

with the Fermi-Dirac distribution function

$$
f(x) = \frac{1}{e^{x/T}+1} \ . \tag{24}
$$

We now first discuss the real case, $N_c = 3$. It is well known that in the absence of isospin degree of freedom the temperature and baryon density effects on chiral symmetry restoration are different [15]: The chiral condensate drops down continuously with increasing temperature, which means a second order phase transition, but jumps down suddenly at a critical baryon density, which indicates a first order phase transition. At finite baryon density and finite isospin density, while the density behavior of the u- and d-quark condensates are different to each other, they may jump down at the same critical baryon chemical potential or at two different critical points. In the case without considering the $U_A(1)$ breaking term, the two critical points do exist, and therefore, there are two chiral phase transition lines [11] in the temperature and baryon chemical potential plane at fixed isospin chemical potential. What is the effect of the $U_A(1)$ breaking term on the QCD phase structure? The calculation in the NJL model showed [13] that if there

exists a structure of two chiral phase transition lines depends on the coupling constant K of the determinant term. Let

$$\alpha = \frac{1}{2}\left(1 - \frac{G-K}{G+K}\right) = \frac{K}{G+K} \ , \tag{25}$$

it was found [13] that at $\mu_I = 60$ MeV the two-line structure disappears for $\alpha > 0.11$. This means that the two-line structure is true only at small ratio α. However, α is a free parameter in [13] and the two coupling constants G and K are not yet separately determined.

Before the discussion of the problem in real world with nonzero current quark mass, it is instructive to analyze the phase structure in chiral limit with $m_0 = 0$ even though pion condensation will happen at finite μ_I. One can regard this as the limit of $m_0 \to 0$. From the gap equations (23) one can clearly see that when one of the quark condensates becomes zero, the other one is forced to be zero for any coupling constants G and $K \neq 0$. Therefore, there is only one chiral phase transition line at any $K \neq 0$ in chiral limit. The two-line structure happens only at $K = 0$. In this case, the two gap equations decouple, the two chiral phase transition lines are determined by

$$\begin{aligned}
&1 - 12G\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{|\mathbf{k}|}\left[1 - f\left(|\mathbf{k}| - \frac{\mu_B}{3} - \frac{\mu_I}{2}\right) - f\left(|\mathbf{k}| + \frac{\mu_B}{3} + \frac{\mu_I}{2}\right)\right] = 0 \ , \\
&1 - 12G\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{|\mathbf{k}|}\left[1 - f\left(|\mathbf{k}| - \left(\frac{\mu_B}{3} - \mu_I\right) - \frac{\mu_I}{2}\right) - f\left(|\mathbf{k}| + \left(\frac{\mu_B}{3} - \mu_I\right) + \frac{\mu_I}{2}\right)\right] = 0 \ ,
\end{aligned} \tag{26}$$

the difference in baryon chemical potential between the two critical points at fixed T and μ_I is

$$\Delta\mu_B(T, \mu_I) = 3\mu_I \ . \tag{27}$$

In real world there are four parameters in the NJL model, the current quark mass m_0, the three-momentum cutoff Λ, and the two coupling constants G and K. Among them m_0, Λ and the combination $G + K$ can be determined by fitting the chiral condensate σ, the pion mass m_π and the pion decay constant f_π in the vacuum. In [13] $m_0 = 6$ MeV, $\Lambda = 590$ MeV, and $(G + K)\Lambda^2/2 = 2.435$, for which $\sigma = 2(-241.5\ \text{MeV})^3$, $m_\pi = 140.2$ MeV, and $f_\pi = 92.6$ MeV. To determine the two coupling constants separately or the ratio α, one needs to know the $\eta\prime$- or a-meson properties in the vacuum.

In the vacuum the two quark condensates degenerate, and the pole equations determining the meson masses are much simplified as

$$\begin{aligned}
&1 - 4N_c(G+K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2 - M_q^2}{E_q^2 - m_\sigma^2/4} = 0 \ , \\
&1 - 4N_c(G-K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2 - M_q^2}{E_q^2 - m_a^2/4} = 0 \ , \\
&1 - 4N_c(G+K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2}{E_q^2 - m_\pi^2/4} = 0 \ , \\
&1 - 4N_c(G-K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2}{E_q^2 - m_{\eta'}^2/4} = 0 \ ,
\end{aligned} \tag{28}$$

with quark energy $E_q = \sqrt{\mathbf{k}^2 + M_q^2}$. It is easy to see that if $K = 0$, the masses of $\sigma-$ and a-mesons degenerate and the masses of η'- and π-mesons degenerate, and if $K = G$ as in the standard NJL Lagrangian, the a- and $\eta\prime$-mesons disappear in the model.

The mass equation for π-meson can be used to determine the combination $G + K$, which is already considered in [13], and the mass equation for $\eta\prime$-meson is related to the combination $G - K$ and then to the ratio

$$\alpha = \frac{1}{2}\left(1 - \frac{1}{G+K}\frac{1}{12\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{E_q}{E_q^2 - m_{\eta'}^2/4}}\right) \ . \tag{29}$$

Combining with the above known parameters m_0, Λ and $G + K$ obtained in [13] and choosing $m_{\eta'} = 958$ MeV, we have $\alpha = 0.29$ which is much larger than the critical value 0.11 [13] for the two-line structure at $\mu_I = 60$ MeV. In fact, for a wide mass region 540 MeV $< m_{\eta'} <$ 1190 MeV, we have $\alpha > 0.11$, and there is no two-line structure.

To fully answer the question if the two-line structure exists before the pion condensation happens, we should consider the limit $\mu_I = \mu_I^c$ where the difference in baryon chemical potential between the two critical points is the maximum, if the two-line structure happens. From the approximate result of lattice simulation [5] and the exact result of NJL analyze in next section, we know that the critical isospin chemical potential for pion superfluidity is equal to the pion mass in the vacuum, $\mu_I^c = m_\pi$. From the analysis above, at this value the isospin chemical potential difference between the two critical points corresponding, respectively, to $\sigma_u = 0$ and $\sigma_d = 0$ in chiral limit and with $K = 0$ is $\Delta\mu_B = 3m_\pi$. In real world with $m_0 \neq 0$ and $K \neq 0$, with increasing coupling constant K or the ratio α the two lines approach to each other and finally coincide at about $\alpha = 0.21$ which is still less than the value 0.29 calculated by fitting $m_{\eta'} = 958$ MeV. Therefore, the two-line structure disappears if we choose $m_{\eta'} = 958$ MeV. In fact, for 720 MeV $< m_{\eta'} <$ 1140 MeV, we have still $\alpha > 0.21$ and the two phase transition lines are cancelled in this wide mass region. When the η'-meson mass is outside this region, the $U_A(1)$ breaking term is not strong enough to cancel the two-line structure, but the two lines are already very close to each other. Considering we have used the η' mass in three flavor world, we should use a smaller η' mass in two flavor world. Using the approximate relation $m_{\eta'}^2 \propto N_f$ one can estimate $m_{\eta'} \approx 780MeV$ which is in the region where $\alpha > 0.21$.

At this stage one may ask in which case the two chiral phase transition lines at finite isospin chemical potential appear. To answer this question we pay attention to the relation between the η' mass and N_c. It is believed the η' mass decreases with increasing N_c [24,25]. At sufficiently large N_c, the $U_A(1)$ breaking induced flavor mixing effect can be neglected and there will be two chiral phase transition lines.

IV. CRITICAL ISOSPIN CHEMICAL POTENTIAL FOR PION CONDENSATION

Since both thermal motion and nonzero baryon number will increase the critical isospin density for pion superfluidity, the minimum isospin chemical potential μ_I^c corresponds to the parameters $T = 0$ and $\mu_B = 0$. In this case the pion condensate is zero and the two quark condensates degenerate, the pole equations for the meson masses are then reduced to

$$
\begin{aligned}
&1 - 4N_c(G+K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2 - M_q^2}{E_q^2 - M_\sigma^2/4} = 0 \ , \\
&1 - 4N_c(G-K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2 - M_q^2}{E_q^2 - M_{a_0}^2/4} = 0 \ , \\
&1 - 4N_c(G-K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2 - M_q^2}{E_q^2 - \left(M_{a_\pm} \pm \mu_I\right)^2/4} = 0 \ , \\
&1 - 4N_c(G+K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2}{E_q^2 - M_{\pi_0}^2/4} = 0 \ , \\
&1 - 4N_c(G+K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2}{E_q^2 - \left(M_{\pi_\pm} \pm \mu_I\right)^2/4} = 0 \ , \\
&1 - 4N_c(G-K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2}{E_q^2 - M_{\eta'}^2/4} = 0 \ ,
\end{aligned}
\tag{30}
$$

where the quark mass M_q satisfies the same gap equation as in the vacuum,

$$
M_q - m_0 - 4N_c(G+K)M_q\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q} = 0 \ , \tag{31}
$$

namely,

$$
M_q(T=0, \mu_B=0, \mu_I \le \mu_I^c) = M_q(0,0,0) \ . \tag{32}
$$

Therefore, from the comparison with the meson mass equations (16) in the vacuum, we derive the isospin dependence of the meson masses for $\mu_I \le \mu_I^c$,

$$
\begin{aligned}
M_\sigma(\mu_I) &= m_\sigma \ , \\
M_{a_0}(\mu_I) &= m_a \ ,
\end{aligned}
$$

$$
\begin{aligned}
&M_{a_\pm}(\mu_I) = m_{a_\pm} \mp \mu_I \ , \\
&M_{\pi_0}(\mu_I) = m_\pi \ , \\
&M_{\pi_\pm}(\mu_I) = m_{\pi_\pm} \mp \mu_I \ , \\
&M_{\eta\prime}(\mu_I) = m_{\eta\prime} \ .
\end{aligned} \tag{33}
$$

We see that the mesons with zero isospin charge keep their vacuum values, the mesons with positive isospin charge drop down linearly in the isospin chemical potential, and the mesons with negative isospin charge go up linearly in the isospin chemical potential. Since $m_a > m_\pi$, the above isospin dependence will firstly break down at $\mu_I = m_\pi$. Beyond this value the mass of π_+ becomes negative and it makes no sense. This gives us a strong hint that the end point of the normal phase without pion condensation or the starting point of the pion superfluidity phase is at $\mu_I^c = m_\pi$.

To prove this hint we should consider directly the isospin behavior of the order parameter of the pion superfluidity, namely the pion condensate. Deriving the matrix elements $\mathcal{S}_{ud}$ and $\mathcal{S}_{du}$ from (8) and then performing the Matsubara frequency summation, the gap equation determining self-consistently the pion condensate as a function of isospin chemical potential at $T = 0$ and $\mu_B = 0$ reads,

$$
\pi \left[1 - 2N_c(G+K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(\frac{1}{\sqrt{\left(E_q - \mu_I/2\right)^2 + \left(G+K\right)^2\pi^2}} + \frac{1}{\sqrt{\left(E_q + \mu_I/2\right)^2 + \left(G+K\right)^2\pi^2}}\right)\right] = 0 \ . \tag{34}
$$

Obviously, the minimum isospin chemical potential μ_I^c where the pion superfluidity starts is controlled by the square bracket with $\pi = 0$,

$$
1 - 4N_c(G+K)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E_q}\frac{E_q^2}{E_q^2 - \left(\mu_I^c\right)^2/4} = 0 \ . \tag{35}
$$

From the comparison of (35) with the pion mass equation (16) in the vacuum, we find explicitly that the critical isospin chemical potential μ_I^c for the pion superfluidity phase transition at $T = \mu_B = 0$ is exactly the vacuum pion mass m_π,

$$
\mu_I^c = m_\pi \ . \tag{36}
$$

We emphasize that this conclusion is independent of the model parameters and the regularization scheme, it is a general conclusion in mean field approximation for quarks and RPA approximation for mesons. Unlike the chiral phase structure which strongly depends on $U_A(1)$ breaking coupling K and color degree of freedom N_c, the critical isospin chemical potential μ_I^C does not depend on them.

V. CONCLUSIONS

We have investigated the two flavor NJL model with $U_A(1)$ breaking term at finite isospin density, as well as at finite temperature and baryon density. With a self-consistent treatment of quarks in mean field approximation and mesons in RPA, we determined the two coupling constants separately by fitting the meson masses in the vacuum, and then investigated the phase structure for chiral symmetry restoration and pion superfluidity, especially the isospin effect on the chiral phase transition line and the minimum isospin chemical potential for pion superfluidity.

The main conclusions are:

1) The two chiral phase transition lines in the $T - \mu_B$ plane predicated by the NJL model without $U_A(1)$ breaking term is cancelled by the strong $U_A(1)$ breaking term at low isospin chemical potential in real world ($N_c = 3$). Only when the isospin chemical potential μ_I is close to the critical value μ_I^c of pion condensation, there is probably the two-line chiral structure, depending on the $\eta\prime$-meson mass. Therefore, in relativistic heavy ion collisions where the typical μ_I value is much less than μ_I^c, it looks impossible to realize the two-line structure. However, it can be realized in large N_c QCD.

2) The critical isospin chemical potential for pion condensation in NJL model is exactly the pion mass in the vacuum, $\mu_I^c = m_\pi$, independent of the model parameters, the regularization scheme, the $U_A(1)$ breaking term and the color degree of freedom N_c.

The temperature and baryon and isospin density dependence of the chiral and pion condensates as well as the meson masses, and the extension to flavor SU_3 NJL model are under way.

Acknowledgments: We thank Dr. Meng Jin for helpful discussions. The work was supported in part by the grants NSFC10135030, SRFDP20040003103 and G2000077407.

---

[1] For instance, see Quark-Gluon Plasma, ed. R.C.Hwa (world Scientific, Singapore, 1990).
[2] M.Alford, K.Rajagopal, and F.Wilczek, Phys. Lett. **B422**, 247 (1998); R.Rapp, T.Schäfer, E.V.Shuryak, and M.Velkovsky, Phys. Rev. Lett. **81**, 53 (1998).
[3] F.karsch, Lect. Notes Phys. **583**, (2002) 209.
[4] D.T.Son and M.A.Stephanov, Phys.Rev.Lett.86, (2001)592 ;Phys.Atom.Nucl.64, (2001)834.
[5] J.B.Kogut,D.K.Sinclair, Phys.Rev.D66, (2002)034505.
[6] J.B.Kogut and D.Toublan, Phys.Rev.D64, (2001) 034007.
[7] M.Loewe and C.Villavicencio, Phys.Rev. D67, (2003)074034; Phys.Rev.D70, (2004) 074005
[8] A.Barducci, R.Casalbuoni, G.Pettini, L.Ravagli, Phys.Lett. B564, (2003) 217.
[9] B.Klein, D.Toublan and J.J.M.Verbaarschot, Phys.Rev. D68, (2003)014009.
[10] Yusuke Nishida, Phys.Rev.D69, (2004)094501
[11] D.Toublan and J.B.Kogut, Phys.Lett. B564, (2003)212.
[12] A.Barducci, R.Casalbuoni, G.Pettini, L.Ravagli, Phys. Rev.D69, (2003)096004.
[13] M.Frank, M.Buballa and M.Oertel, Phys.Lett. B562, (2003)221.
[14] Y.Nambu and G.Jona-Lasinio, Phys. Rev. **122** (1961) 345; **124** (1961)246.
[15] For reviews and general references, see U.Vogl and W.Weise, Prog. Part. and Nucl. Phys. **27** (1991) 195; S.P.Klevansky, Rev. Mod. Phys. **64** (1992) 649; M.K.Volkov, Phys. Part. Nucl. **24** (1993) 35; T.Hatsuda and T.Kunihiro, Phys. Rep. **247** (1994)338.
[16] J.Hufner, S.P.Klevansky, P.Zhuang and H.Voss, Ann.Phys(N.Y) 234, (1994)225; P.Zhuang, J.Hufner and S.P.Klevansky, Nucl.Phys.A576, (1994)525.
[17] P.Zhuang and Z.Yang, Phys.Rev.**C62**,(2000)054901.
[18] T.M.Schwarz, S.P.Klevansky, and G.Rapp, Phys. Rev. **C60** (1999)055205.
[19] M.Huang, P.Zhuang, and W.Chao, Phys.Rev.**D65**, (2002)076012.
[20] M.Huang, P.Zhuang, and W.Chao, Phys.Rev.**D67**, (2003)065015.
[21] J.Liao, and P.Zhuang, Phys. Rev. **D68**, (2003)114016.
[22] I.Shovkovy, and M.Huang, Phys. Lett. **B564**, (2003)205.
[23] G.t'Hooft, Phys.Rev.D 14(1976)3432;Phys.Rep.142(1986)357
[24] E.Witten, Nucl.Phys.B149,285(1979)
[25] T.Schäfer, Phy.Rev.D66, 076009(2002);Phys.Rev. D67, 074502(2003)